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Anomalous Dynamics in the Ising Chain

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We analyze a 1D Ising system with anomalous distributions of nearest neighbor interactions and show that the single-spin-flip dynamics exhibit breakdown of dynamic scaling. The results are obtained by a real-space numerical method applied to the exact equations of motion and they may be explained by domain wall motion arguments reformulated in terms of extreme value statistics.

KEY WORDS: Renormalization group; Glauber dynamics; Ising chain; critical dynamics; anomalous dynamics; breakdown of dynamic scaling; dynamic correlation length.

1. INTRODUCTION

According to the restricted dynamic hypothesis,[1] the characteristic time τ evolution of the order parameter scales with correlation length ξ according to the law

$$\tau \sim \xi^{Z} \tag{1}$$

where the critical dynamical exponent Z is expected to depend only on some universal features of the underlying model (e.g., space and spin dimensionality, conservation laws, etc).

However, it was recently recognized that universality fails in critical dynamics at 1D: Z is found[2,3] to depend on the distribution of interactions in the system. For randomly distributed, nearest neighbor interactions $J_{i,i+1}$ with discrete but finite values, one finds

$$Z = 1 + J_M/J_m \tag{2}$$

where $J_M = \text{Max}\{J_{i,i+1}\}$ and $J_m = \text{Min}\{J_{i,i+1}\}$.

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However, even the scaling law expressed by Eq. (1) breaks down[4-6] for the dilute Ising model at the percolation threshold, where one obtains a dependence of Z on the temperature:

$$Z = A \ln \xi + B \tag{3}$$

The present work numerically analyzes a 1D Ising model that interpolates between the preceding two cases: nonuniversality and breakdown of dynamic scaling. We consider two extreme situations of the model leading to Eq. (2):

(i) Case A. $J_m = 0$ and the anomalous distribution $P(J) = (1-\alpha) J^{-\alpha}$, $0 < J \leqslant 1$, $0 < \alpha < 1$. In this case

$$\xi^{-1} = -\langle \ln \operatorname{th} \beta J \rangle \sim \beta^{-(1-\alpha)} \tag{4}$$

where $\beta = 1/K_B T$ and T is the temperature.

(ii) Case B. $J_M = \infty$ and the anomalous distribution $P(J) = (1-\alpha) J^{-2+\alpha}$, $1 \leqslant J < \infty$, $0 < \alpha < 1$. In this case

$$\xi \sim \beta e^{2\beta} \tag{5}$$

As in refs. 2 and 3, dynamics is introduced in the Glauber way[7] by defining a single-spin flip rate

$$W_i(\sigma_i) = 1/2 w_i [1 - \sigma_i(\gamma_i^- \sigma_{i-1} + \gamma_i^+ \sigma_{i+1})]$$

where σ_i denotes the flipping spin ($\sigma_i = \pm 1$) and

$$\gamma_i^\pm = 1/2[\operatorname{th}(k_{i,i+1} + k_{i-1,i}) \pm \operatorname{th}(k_{i,i+1} - k_{i-1,i})]$$

with $k_{i,i+1} = J_{i,i+1}/K_B T$. We assume uniform intrinsic rates w_i for cases A and B.

The particular situation with homogeneous interactions $J_{i,i+1} = J$ and an anomalous distribution of intrinsic flipping rates $P(w_i) = (1-\alpha) w_i^{-\alpha}$, $0 < \alpha < 1$, $0 < w_i \leqslant 1$, has already been treated analytically.[8] The nonuniversal result

$$Z = (2-\alpha)/(1-\alpha)$$

is reconfirmed here and it agrees with well-known random walk arguments.[9]

For the cases A and B, the usual domain wall argument[10] has to be reformulated. For a completely random, unbiased distribution of exchanges, the wall is still expected to perform a random walk. The time to displace through a domain length (typically, ξ) will therefore be

$$\tau \sim \xi^2 \tau_{\mathrm{MAX}}$$

Here, τ_{MAX} is the maximum time step (within ξ) for wall motion; this is expected to be given by an activation law

$$\tau_{\mathrm{MAX}} \sim \exp\{2\beta[J_M(\xi) - J_m(\xi)]\} \tag{6}$$

where $J_M(\xi)$ $[J_m(\xi)]$ is the greatest (smallest) exchange which the wall meets on a length scale ξ. Accordingly, we find:

(i) Case A:

$$\tau \sim \xi^2 e^{2\beta}$$

or, using Eq. (4), we obtain

$$\log \tau \sim \xi^{1/(1-\alpha)} \tag{7a}$$

(ii) Case B:

$$\tau \sim \xi^2 e^{2\beta J_M(\xi)}$$

where $J_M(\xi) \sim \xi^{1/(1-\alpha)}$. Using Eq. (5), we obtain

$$\log \tau \sim \xi^{1/(1-\alpha)} \log \xi \tag{7b}$$

We thus expect strong violations of the scaling law; in case A, they arise from the anomalous thermal dependence of the correlation length (dominated by weak bonds), and, in case B, from arbitrarily large energy barriers found by the wall in its motion.

To check these predictions, we now consider an exact formulation of dynamics and we shall use numerical renormalization group methods to extract the long-time behavior.

2. BASIC THEORY

Following the stochastic formulation of Glauber,[7] the exact equations of motion for the mean value of σ_i $(M_i = \langle \sigma_i \rangle_t)$ may be written as

$$d/dt\, M_i = w_i(-M_i + \gamma_i^- M_{i-1} + \gamma_i^+ M_{i+1}) \tag{8a}$$

Taking the Laplace transform

$$M_i(s) = \int_0^{+\infty} e^{-st} M_i(t)\, dt$$

Eq. (8a) reads

$$M_i(s) = \Gamma_i^- M_{i-1}(s) + \Gamma_i^- M_{i+1}(s) + M_i(t=0)/(w_i + s) \tag{8b}$$

where $\Gamma_i^{\pm}(s) = \gamma_i^{\pm}/(1 + s/w_i)$.

The inhomogeneous terms in Eq. (8b), associated with the initial condition $M_i(t=0)$, are irrelevant since the relaxation times for the system are determined by the eigenvalues of the matrix associated with the corresponding homogeneous equation. We observe that the above equations still hold (for $i \neq 0$) if we fix the spin at the origin through, e.g., application of a sufficiently strong magnetic field (the equation for $i=0$ is, of course, different from the above). In this case, $M_i(t)$ represents the time-dependent, averaged magnetization conditioned to a fixed magnetization at the origin—that is, $M_i(t)$ directly expresses the time-dependent two-spin correlation and this is expected to show exponential space decay, not only at equilibrium ($s=0$), but also away from equilibrium. We shall show that these expectations are fulfilled, and may therefore introduce a dynamic correlation length, which goes over to the usual thermodynamic one in the static ($s=0$) limit.

We now apply real-space renormalization-group techniques to deal with Eq. (8b). By decimating every odd spin, the new equations are formally identical to the old ones, but with renormalized parameters

$$\Gamma_i'^{\pm} = \frac{\Gamma_{2i}^{\pm}\Gamma_{2i\pm 1}^{\pm}}{1 - \Gamma_{2i}^{+}\Gamma_{2i+1}^{-} - \Gamma_{2i}^{-}\Gamma_{2i-1}^{+}} \tag{9}$$

After successive iteractions, each one doubling the lattice spacing, the origin will have at the rescaled lattice a nearest neighbor which, in the original lattice, is many lattice spacings apart. The iterated parameters $\Gamma_i^{\pm}$ will therefore envolve toward the averaged two-spin correlation function and this will decay exponentially (at least at large distances, in the inhomogeneous lattice, and for the static case).

In the homogeneous case, Eq. (9) reads

$$\Gamma' = \Gamma^2/(1 - 2\Gamma^2)$$

or

$$\chi'^{-1} = 2\chi^{-1} \tag{10}$$

with $\Gamma = 1/(2\,\mathrm{ch}\,\chi^{-1})$. For $T \sim T_c = 0$, $s \sim 0$,

$$\chi^{-1}(\xi^{-1}, s) \sim (s + \xi^2)^{1/2} \sim \xi^{-1}(1 + \tfrac{1}{2}s\xi^2)$$

that is, after n $(2^n \gg \xi)$ iterations

$$\Gamma^{(n)} \sim e^{-2^n \chi^{-1}(\xi^{-1}, s)} \tag{11}$$

In this case, we may therefore interpret χ as a dynamic correlation length.

In the limit $\xi^{-1} = 0$, Eq. (10) implies

$$\chi^{-1}(s) \sim s^{1/z} \qquad \text{with} \quad z = 2 \tag{12}$$

We generalize to the disordered case the preceding scaling behavior (10), (11) by stating as an ansatz that $\chi_n/b^n \to \chi$ as $n \to \infty$, where $\chi(\xi^{-1}, s)$ obeys the scaling form

$$\chi^{-1}(b\xi^{-1}, b^z s) = b\chi^{-1}(\xi^{-1}, s) \tag{13}$$

where $b = 2$ is the lattice rescaling factor.

Moreover, we smooth out inhomogeneities, taking for χ_n an average defined by

$$\frac{b^n - 1}{N} \sum_{i=1}^{N/(b^n - 1)} \Gamma_i^{+(n)} \equiv 1/(2 \operatorname{ch} \chi_n^{-1}) \tag{14}$$

3. RESULTS

We use a numerical renormalization of a chain of $N = 2^{15}$ spins successfully tested in a previous case,[2,3] where a random, discrete distribution of interactions with finite values was used. This procedure, similar to that of refs. 11 and 12, consists in first generating the chain according to the relevant distribution. Next, the system is renormalized using Eq. (9) in successive iterations; and finally, an extrapolation of $\chi^{-1}(\xi^{-1}, s)$ according to Eq. (14) is obtained. An average over 10 realizations (100 in case B) of each system is also performed.

Case A required about 4 hr of CPU on an APPLE-MAC II microcomputer for each value of α.

We use the well-known random number generator[13]

$$X_i = X_{i-24} + X_{i-55} \pmod 1$$

that permits a cycle of more than 2^{55}. This algorithm is initialized by generating the first $50 X_i$ $(0 < X_i \leqslant 1)$ using any simpler generator (or the system's).

Our main results follow.

(i) In the case of homogeneous interactions and w_n anomalously distributed, one can set $T=0$ and, using Eq. (13), obtain

$$\chi^{-1}(s) \sim s^{1/z}$$

This fits data very well, as is illustrated (for $\alpha=1/2$) in Fig. 1. Using other values of α, we checked the result $z=(2-\alpha)/(1-\alpha)$.[6]

(ii) In case A, we may cast Eq. (13) in the form

$$\chi^{-1}/\xi^{-1} = f(s\tau) \tag{15}$$

$\tau=\tau(\xi)$ and $f(x)$ is a scaling function. For $s\tau \ll 1$

$$\chi^{-1}/\xi^{-1} - 1 \sim s\tau(\xi)$$

This was used to extract $\tau(\xi)$ as is shown in Fig. 2 for $\alpha=1/2$. A plot of $\log\tau$ versus β permits us to confirm the scaling relation of Eq. (7a): for $\alpha=1/2$ (Fig. 3) such a dependence is confirmed within 2%.

Therefore, we obtain

$$z \sim \beta/\log\beta \tag{16}$$

which is manifestly a consequence of the anomalous dependence of Eq. (4) of ξ on the temperature.

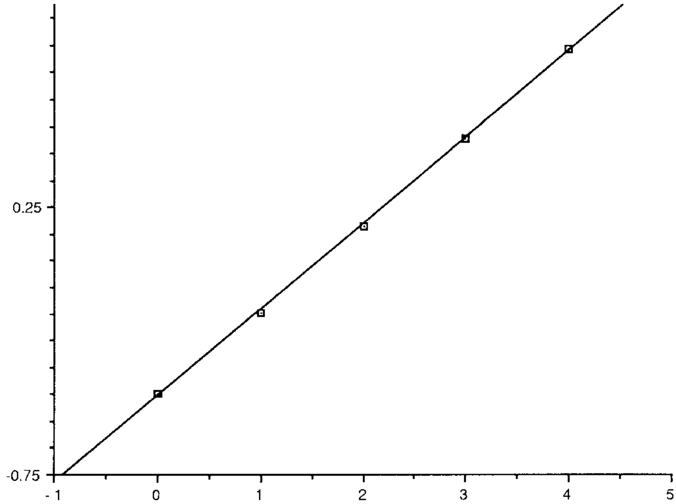

Fig. 1. Case of anomalously distributed intrinsic flipping rates ($\alpha=1/2$): $\log\tau$ versus $\log s$. It is found that $1/z=0.32$.

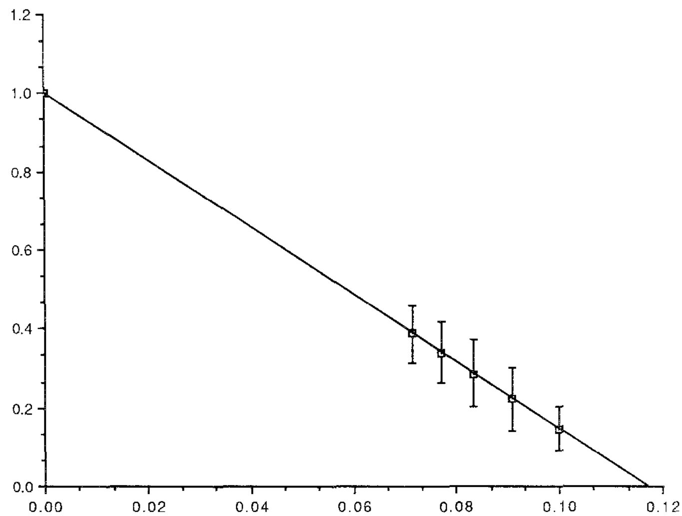

Fig. 2. Case A ($\alpha = 1/2, \beta = 12$): $\log(\chi^{-1}/\xi^{-1} - 1)/(-\log s)$ versus $-1/\log s$. The slope represents $\log \tau$.

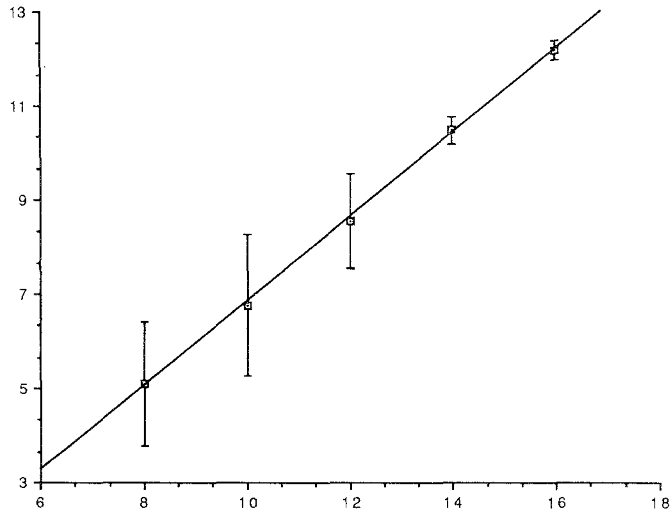

Fig. 3. Case A ($\alpha = 1/2$): $\log \tau$ versus β. It is found that $\tau \sim e^{b\beta}$, with $b = 2.05$.

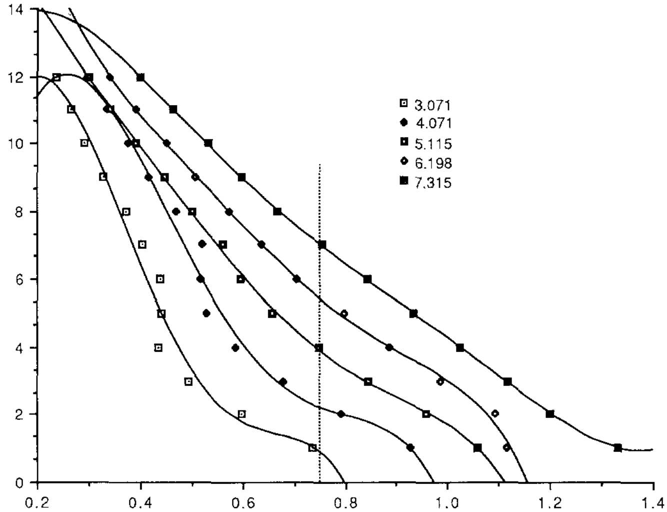

Fig. 4. Case B $[1/(1-\alpha)=1.1]$: $-\log s$ versus χ^{-1}/ξ^{-1}. The labels refer to the values of ξ, and the vertical bar at $\chi^{-1}/\xi^{-1}=0.75$ corresponds to the a fixed value $x=s\tau$ for which we read from the graph the value of $\log \tau(\xi)$.

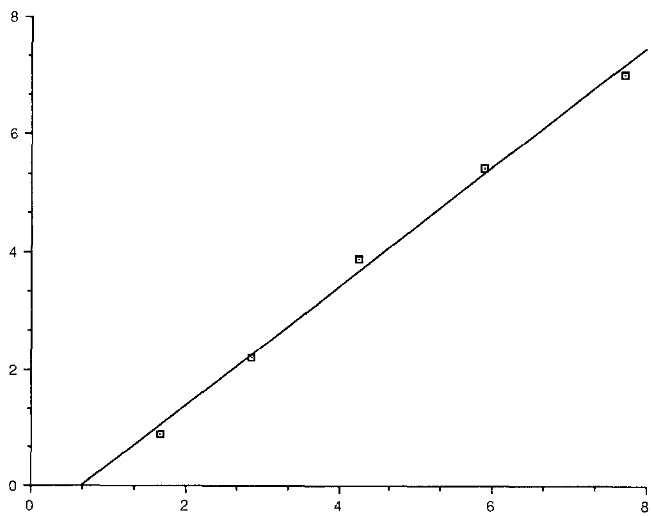

Fig. 5. Case B $[1/(1-\alpha)=1.1]$: $\log \tau$ versus $\xi^{1.1} \log \xi$.

(ii) For case B we use the full scaling form (15), as illustrated in Fig. 4: we extract $\log \tau$ by first fixing $x = s\tau(\xi)$ and varying ξ. In Fig. 5 we obtain a good fitting for Eq. (7b).

It must be emphasized that the dependence expressed by Eq. (7b), contrary to case A, comes purely from the dynamics of the model: the existence of very "slow spins" with a flipping rate that goes as $\exp[-2\beta\, |J_M(\xi) - J_m(\xi)|]$ over a length ξ, as expressed by Eq. (6).

These results must be confronted with the domain wall argument of ref. 3:

$$\tau \sim \xi^2 \langle e^{-2\kappa_{i,i+1}} \rangle \langle e^{+2\kappa_{i,i+1}} \rangle \tag{17}$$

In case A this gives essentially the dependence of Eq. (7a):

$$\tau \sim e^{2\beta}/\beta^{\alpha}$$

However, for case B, Eq. (17) fails to hold, and the mean values have to be interpreted in terms of extreme value statistics over a length ξ giving Eq. (7b).

4. CONCLUSIONS

In conclusion, we have tested our method and hypotheses in the case with anomalous randomness,[6] for which an analytic result is known.

We analyzed a 1D system where the restricted dynamic scaling hypothesis breaks down, leading to a temperature-dependent dynamic exponent, as in Eq. (16). We also have shown the existence of a situation (case B) where the usual domain wall argument[10] has to be reformulated in terms of the statistics of extreme values over a length ξ.

Our analysis supports the concept of a dynamic correlation length $\chi(\xi^{-1}, s)$ which, for a given frequency s, describes the decay of spin correlations and goes over to the equilibrium correlation length ξ in the static limit ($s = 0$).

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